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The brachistochrone problem for a disc☆

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ABSTRACT

The motion of a vertical disc along a curve under the influence of gravity is investigated. On the assumption of regular rolling without slip and separation of contact points, the problem of plotting the curve of most rapid motion of the disc centre from the origin of coordinates to an arbitrary fixed point of the lower half-plane is solved. As usual, the velocity at the initial instant of time is zero, and at the final instant of time it is not fixed. In explicit parametric form, the classical brachistochrone for contact points of the disc is plotted and investigated. The response time, trajectory and kinematic and dynamic characteristics of motion are calculated analytically. Previously unknown qualitative properties of regular rolling are established. In particular, it is shown that the disc centre moves along a cycloid connecting specified points. The envelopes of the boundary points of the disc, produced as its centre moves along the cycloid, are brachistochrones. The feasibility of mechanical coupling of the disc and the curve by reaction forces at the contact point (the normal pressure and dry friction) is investigated.

1. Formulation of the problem

The rolling of a dynamically symmetrical, possibly inhomogeneous, vertical disc plane along a fixed curve is considered (Fig. 1). The shape of the curve is assumed to be fairly smooth and is selected on the basis of additional conditions. Motion is considered to occur without slip and separation. These requirements can be satisfied with appropriate properties of contact (the roughness of the interacting parts), the curvature of the line (the magnitudes of the radii of curvature) and the conditions of motion (the phase variables). The reaction forces (the normal pressure and sliding friction) are determined from the equations of motion (see below). The process of regular rolling can also be realized artificially by additional normal constraints, not preventing motion. Rolling friction, governed by deformation, is not considered in order to keep the investigation simple.

Owing to the specific nature of the problem, a more convenient left-hand system of coordinates *OXY* is traditionally introduced. The *Y* axis is assumed to be directed vertically downwards along the directrix of the force of gravity. By analogy with the classical brachistochrone problem for a point mass (see Refs 1–3, etc.), the problem of plotting the curve of most rapid regular rolling of a disc from a fixed state of rest to a required position at an arbitrary velocity is formulated. The points defining the positions of the disc centre *C* at the initial instant ($t = 0$) and the final instant ($t = t_f$) of time are considered to be specified. Without loss of generality, the origin of coordinates *O*, i.e., $X = Y = 0$, is adopted as the initial point and the point *D* with coordinates $X = x_c^f$, $Y = y_c^f$ is adopted as the final point. Obviously, to be specific it is essential that the ordinate $y_c^f > 0$, and the abscissa $x_c^f > 0$, i.e., trajectories of rolling to the right are considered (see Fig. 1).

The variational problem is as follows: it is required to construct the rolling curve *AB*, i.e., the function $y(x, A, B)$, such that, under the action of the forces and the moments of forces of gravity, the disc centre *C* moves from the initial point *O* to the final point *D* in the shortest time t_f; the initial velocity of the disc centre is zero, and the final velocity is not fixed. The curve must be determined in explicit form $y(x, A, B)$ or implicit form $K(x, y, A, B) = 0$, and here the coordinates of the points *A* and *B* must also be calculated. Usually, the problem can be solved successfully in parametric form $x(\xi, A, B)$, $y(\xi, A, B)$, where ξ is a parameter of the curve, $\xi_0 \le \xi \le \xi_f$; the unknown quantities ξ_0 and ξ_f are determined from the final conditions.

As is well known (see Refs 1–3, etc.), in the classical brachistochrone problem for a point mass sliding without friction (J. Bernoulli, 1696), the solution is a cycloid (Fig. 2), the equation of which in parametric form is as follows:

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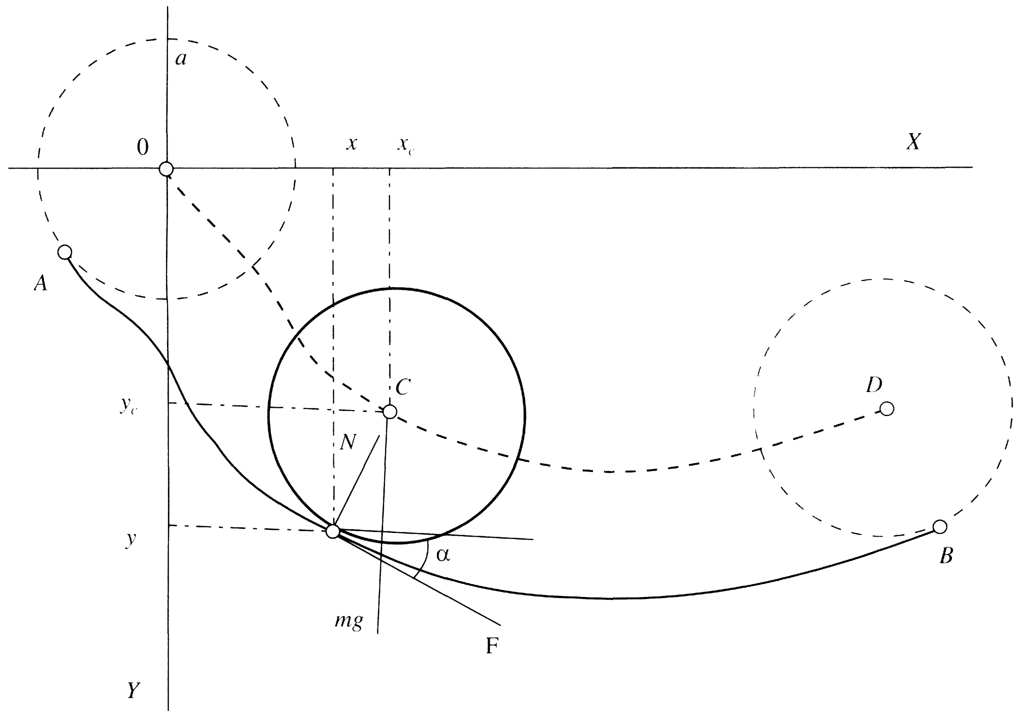

Fig. 1.

$$
\begin{aligned}
&x = R(\xi - \sin\xi), \quad y = R(1-\cos\xi), \quad x = x_c, \quad y = y_c \\
&x_f/y_f \equiv \chi_f = (\xi_f - \sin\xi_f)(1-\cos\xi_f)^{-1}, \quad R = y_f(1-\cos\xi_f)^{-1} \\
&0 = \xi_0 \le \xi \le \xi_f(\chi_f) \le 2\pi, \quad \xi = (g/R)^{1/2}t, \quad 0 \le t \le t_f
\end{aligned}
\tag{1.1}
$$

According to relations (1.1), it is first necessary to find the unknown final value of the parameter $\xi_f(\chi_f)$, and then, from first elementary equations for $y(\xi_f)=y_f$ or $x(\xi_f)=x_f$, the constant R_f is calculated as a function of y_f, ξ_f or x_f, ξ_f, i.e., $R_f(x_f, y_f)$. The dependence of ξ_f on χ_f is specified analytically by the elementary curve $\chi(\xi)$ according to relations (1.1) (see Fig. 3). For the specified value $\chi=\chi_f$, the quantity $\xi=\xi_f$ is found numerically or graphically.

Note that the limiting quantity $\xi_f=2\pi$, corresponding to $y_f=0$, is also achievable. The required time of motion in this case is finite and is calculated from the formula

$$t_f = 2\pi\left(\frac{R_f}{g}\right)^{1/2},$$

where

$$R_f = \left(\frac{x_f}{2\pi}\right).$$

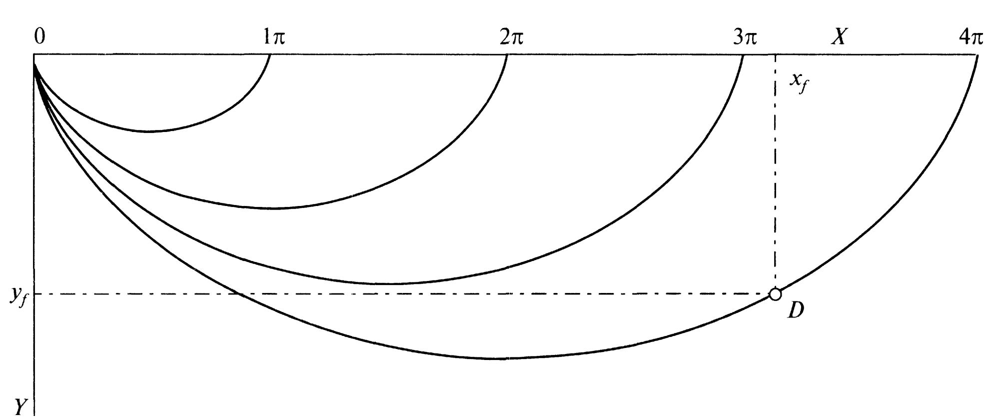

Fig. 2.

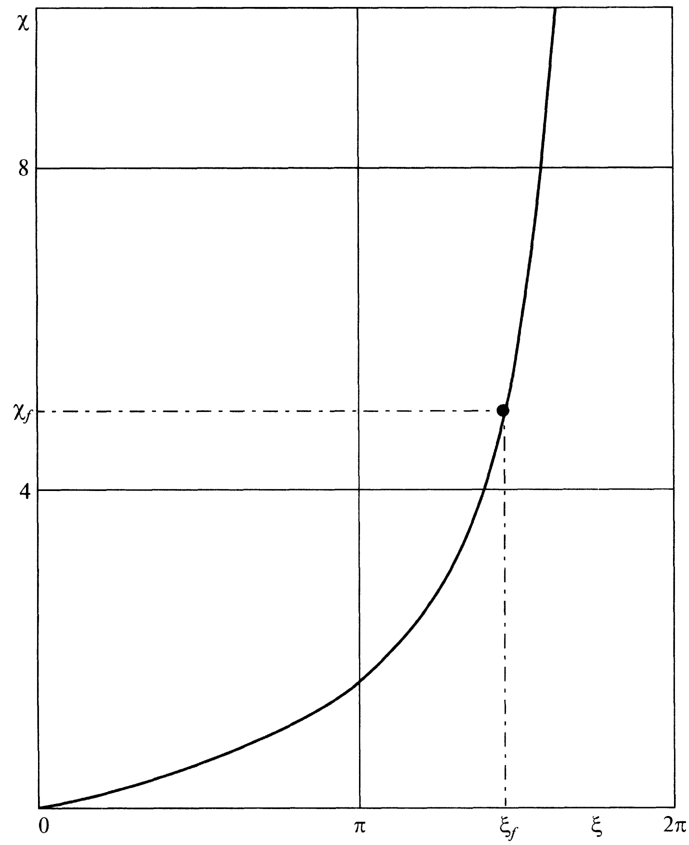

Fig. 3.

Note, in particular, that, as a convenient parameter for setting the curve, we can take its length s ($s'^2 = x'^2 + y'^2$, where a prime denotes a derivative with respect to the parameter ξ), $0 \le s \le s_f$. In this case, the equation of the curve is specified in so-called natural form[4] (see below). After determining the parameters ξ_f and R_f from relations (1.1), the variable ξ is found extremely simply as a function of the natural parameter s – the length of the arc of the curve

$$\xi(s, x_f, y_f) = 2\arccos\left(1 - \frac{s}{4R_f}\right), \quad 0 \le \xi \le \xi_f$$
$$0 \le s \le s_f = 4R_f\left(1 - \cos\frac{\xi_f}{2}\right); \quad \xi = \left(\frac{g}{R_f}\right)^{1/2} t \tag{1.2}$$

By means of expressions (1.2), the required dependences of x, y and s on the time t, $0 \le t \le t_f$, are determined. The minimum value of the functional of the variational problem is calculated in an elementary way by means of the formula

$$t_f = \frac{1}{\sqrt{2g}} \int_0^{s_f} \frac{ds}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \int_0^{\xi_f} \frac{(x'^2 + y'^2)^{1/2}}{\sqrt{y}} d\xi = \left(\frac{R_f}{g}\right)^{1/2} \xi_f \tag{1.3}$$

Here, the values of the parameters ξ_f and R_f are determined using to formula (1.1).

Note that relations (1.1)–(1.3) fully describe the solution of the classical brachistochrone problem – a smooth curve of the most rapid fall of a point mass from a state of rest O at $t = 0$ to a required position D (without fixing the velocity) at $t = t_f$.

By means of formulae (1.1) and (1.2), expressions are obtained for the velocities and accelerations

$$\dot{x}(t) = (gR_f)^{1/2}(1 - \cos\xi), \quad \dot{y}(t) = (gR_f)^{1/2}\sin\xi$$
$$\dot{s}(t) = \upsilon(t) = (\dot{x}^2 + \dot{y}^2)^{1/2} = 2(gR_f)^{1/2}\sin\frac{\xi}{2} \tag{1.4}$$

$$\ddot{x}(t) = g\sin\xi, \quad \ddot{y}(t) = g\cos\xi, \quad w(t) = (\ddot{x}^2 + \ddot{y}^2)^{1/2} = g, \quad \ddot{s} = \dot{\upsilon} = g\cos\frac{\xi}{2} \tag{1.5}$$

where

$$\xi = \xi(t) = \left(\frac{g}{R_f}\right)^{1/2} t, \quad 0 \le t \le t_f, \quad 0 \le \xi \le \xi_f \le 2\pi$$

From expressions (1.4) it follows that $\dot{x} > 0$ when $0 < t \le t_f$, i.e., the abscissa x increases monotonically; the velocity $\dot{x}$ is a maximum when $\xi = \pi$, i.e., on the lower part of the cycloid. According to expressions (1.5), $|\ddot{x}| = g$, i.e., the magnitude of the acceleration is a maximum at $\xi = \frac{\pi}{2}, \frac{3\pi}{2}$; $\ddot{x} = 0$ when $\xi = 0, \pi, 2\pi$; the behaviour of $\ddot{y}$ is the reverse. It is interesting that the velocity profile $\dot{y}$ along the ordinate axis y repeats the acceleration profile $\ddot{x}$ along the abscissa axis x. The velocity modulus υ of the point as it slides along the cycloid increases when $0 < \xi < \pi$, and then decreases when $\pi < \xi < 2\pi$ according to expression (1.5) for $\ddot{s} = \dot{\upsilon}$. The vector components of the acceleration $(\ddot{x}\,\ddot{y})$ (1.5) have the form of projections of the vector of the acceleration due to gravity $(0, g)$ onto the rotating axes with an angle of rotation ξ, $0 \le \xi \le 2\pi$; its modulus $w = g$.

By means of standard formulae[4] based on relations (1.4) and (1.5), the polar coordinates and radial and transverse components of the velocity and acceleration are calculated, and also the tangential and normal components of the acceleration of a point moving along the cycloid under gravity.

The model of sliding of the point mass is considered to be approximately applicable to the problem of the falling of a body of “small” size (by comparison with the minimum radius of curvature). Such an assumption requires us to substantiate the similarity of the solutions for these models. As it slides, the body must be supported at certain points; variable constraints are imposed upon it. By comparison with the translational motion of a point, its motion becomes more complex (translational-rotational). The equations of motion and the relations for the brachistochrone are made far more complicated. Note that the radius of curvature of the cycloid tends to zero as $\xi \to 0, 2\pi$, while the curvature coefficient becomes unbounded (see below).

More serious difficulties arise when plotting the brachistochrone for a rolling body, in particular for a disc. Full systematic investigations and the results of the solution of the classical brachistochrone problem for a disc are not given in the literature.

2. The equations of the regular rolling of a disc

Consider a geometrically smooth curve $y(x)$ passing through the points A and B (see Fig. 1). It is assumed that a disc of radius a can roll in a regular manner along the curve, i.e., without slip and separation.[5,6] If the curvature κ of the curve on certain sections is negative (downward convexity), which occurs for the brachistochrone, then its absolute value does not exceed a^{-1} (see below). At the contact point x, y, first-order tangency occurs.

The position of the disc centre C (and its centre of mass) has the coordinates x_c, y_c[6]

$$x_c = x + a\sin\alpha, \quad y_c = y - a\cos\alpha, \quad y = y(x)$$
$$\operatorname{tg}\alpha = y'(x), \quad \sin\alpha = \frac{y'}{s'}, \quad \cos\alpha = \frac{1}{s'}, \quad s' = (1 + x'^2)^{1/2} \tag{2.1}$$

The angle α is measured clockwise from the X axis (see Fig. 1). If the function $x = x(t)$ is specified and is continuous in time $t > 0$, then the variable $y = y(x(t))$ will also be continuous for $|\alpha| < \pi/2$, which is also assumed. The variables x_c, y_c, however, are defined in terms of $y'(x)$, which will require continuous differentiability of $y(x)$. The kinematic characteristics of the disc centre C are calculated simply by formal differentiation of expressions (2.1) for x_c, y_c with respect to t.[6] For the velocity components we will obtain

$$\dot{x}_c = \dot{x} + a\dot{\alpha}\cos\alpha = (1 + a\kappa)\dot{x}$$
$$\dot{y}_c = \dot{y} + a\dot{\alpha}\sin\alpha = y'(1 + a\kappa)\dot{x} = y'\dot{x}_c; \quad y = y(x), \quad \kappa = y''(1 + y'^2)^{-3/2} \tag{2.2}$$

As noted above, relations (2.2) impose constraints on the quantity κ: $\kappa > -a^{-1}$. When $\kappa \downarrow -a^{-1}$, the relationship between $\dot{x}_c$, $\dot{y}_c$ and $\dot{x}$ degenerates, which leads to essential singularity in the equations of motion.

We will determine the full angular velocity of rotation of the disc for regular rolling along the curve.[6] The angle of complete rotation ψ is governed by the path length s and the rotation of the tangent, i.e., by the angle α:

$$\psi(x) = \varphi(x) + \Delta\alpha(x), \quad \varphi(x) = \frac{s(x)}{a}, \quad \Delta\alpha(x) = \alpha(x) - \alpha^0 \tag{2.3}$$

Here, α^0 is a certain constant, for example $\alpha^0 = \alpha(x^0)$, where x^0 corresponds to $x_c = 0$. The angular velocity ψ is also found by formal differentiation of expressions (2.3) taking into account relations (2.1)

$$\dot{\psi} = \dot{\varphi} + \dot{\alpha} = s'a^{-1}(1 + a\kappa)\dot{x}, \quad \dot{\varphi} = s'a^{-1}\dot{x}, \quad \dot{\alpha} = s'\kappa\dot{x} \tag{2.4}$$

From relations (2.2) and (2.4) it follows that we have continuity of the characteristics $\dot{x}_c$, $\dot{y}_c$ and $\dot{\psi}$ with respect to t provided there is continuity of the curvature function κ, i.e. y''. This condition imposes higher continuity requirements on the class of functions $y(x)$ considered in the variational problem compared with the classical brachistochrone (for a point mass). Calculation of the accelerations $\ddot{x}_c$, $\ddot{u}_c$ and $\ddot{\psi}$ will require continuity (or differentiability in order to satisfy the conditions of the existence and uniquely theorem) of the function κ', i.e., the function y'''.

The variable x will be taken as the generalized coordinate for describing the regular rolling of the disc in a gravity field. The remaining geometric and kinematic characteristics of the system must be expressed in terms of x and $\dot{x}$, enabling us to determine the kinetic energy

$T(x, \dot{x})$ and the potential energy $V(x)$. We have the expressions[6]

$$T = \frac{m}{2}(\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2}I\dot{\psi}^2 = \frac{1}{2}\mu(x)\dot{x}^2, \quad V = -mgy_c = -mg\left(y - \frac{a}{s'}\right)$$
$$\mu = m^*(1 + a\kappa)^2 s'^2, \quad m^* = m + \frac{I}{a^2}, \quad \kappa = \frac{y''}{s'^3} \tag{2.5}$$

where μ is the generalized inertia characteristic of the rolling disc, m^* is the reduced mass and I is the central moment of inertia. Note that representation (2.5) for the kinetic energy T degenerates when $\kappa \downarrow -a^{-1}$. This singularity imposes the requirement $\kappa(x) > -a^{-1}$.

The equation of motion in the form of the Lagrange's equation has the form[6]

$$\mu(x)\ddot{x} + \mu'(x)\frac{\dot{x}^2}{2} + V'(x) = 0 \tag{2.6}$$

For Eq. (2.6) the Cauchy problem is still not specified. First it is necessary to find the function $y(x)$ from the class of fairly high continuity so that the disk centre C at zero initial velocity moves from point O to point D in the minimum possible time t_f.

Note that for the curve $y(x)$ the initial point $A(x(0) = x^0, y(x^0) = y^0)$ and the final point $B(x(t_f) = x^f, y(x^f) = y^f)$ are unknown and are determined by taking expressions (2.1) into account. Thus, the variational problem is required: the boundary conditions are specified for the displaced curve $x_c(t)$, $y_c(t)$ (2.1) along which the disc centre C is moving. The brachistochrone – the rolling curve $x(t)$, $y(t)$ with unknown endpoints – is plotted. It is not clear in advance which point of the edge of the disc is in contact with the line of rolling.

3. Formulation of the variational problem for the rolling curve

In the standard manner, we will apply the formula for the integral of the total energy E of the conservative system described by Eq. (2.6). Using expressions (2.5), we obtain the governing relation between x and t of differential form

$$E = T + V = \frac{1}{2}\mu(x)\dot{x}^2 - mg\left(y(x) - \frac{a}{s'(x)}\right) = \text{const}$$
$$\mu = m^*(1 + a\kappa(x))^2 s'^2(x), \quad \kappa = \frac{y''(x)}{s'^3(x)}, \quad s' = (1 + y'^2(x))^{1/2} \tag{3.1}$$

From relations (3.1) we have the required integral relation between the variables t and x (the second integral)

$$\theta \equiv \left(2g\frac{m}{m^*}\right)^{1/2} t = \int_{x^0}^{x} \Phi(y, y', y'')dq, \quad \Phi \equiv (1 + a\kappa(x))s'(x)\left(y - \frac{a}{s'(x)}\right)^{-1/2} \tag{3.2}$$

Note, however, that in relations (3.1) and (3.2) the parameters E, $x^{0,f}$ and $y^{0,f}$ and, finally, the shape of the $y(x)$ curve are unknown. It is necessary to formulate the closed variational problem that enables us to determine the required quantities uniquely. From the above we have the relations

$$\theta_f = \int_{x^0}^{x^f} \Phi(y, y', y'')dx \to \min_y \tag{3.3}$$

$$x_c^0 = x^0 + \frac{ay'(x^0)}{s'(x^0)} = 0, \quad y_c^0 = y^0 - \frac{a}{s'(x^0)} = 0$$
$$x_c^f = x^f + \frac{ay'(x^f)}{s'(x^f)}, \quad y_c^f = y^f - \frac{a}{s'(x^f)} \tag{3.4}$$

From the second equation of system (3.4) it follows that the integral $E = 0$. The coordinate $y^0 = a(1 + y_0'^2)^{-1/2}$, where the derivative $y_0' = y'(x_0)$ is also unknown and to be determined. The integrand Φ does not depend on variable x. It is linear in y', which leads to degeneracy of the Euler–Poisson equation:

$$\frac{\partial\Phi}{\partial y} - \frac{d}{dx}\frac{\partial\Phi}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial\Phi}{\partial y''} = 0; \quad \frac{\partial^2\Phi}{\partial y''^2} \equiv 0 \tag{3.5}$$

The stricter Legendre-type condition is not satisfied. It can be established directly by differentiation that third-order equation (3.5) occurs for $y(x)$ rather than a fourth-order equation, as in the general case. This general solution will depend on three integration constants. It should satisfy four boundary conditions containing six unknowns, which makes their determination difficult. These conditions must be predetermined on the basis of additional assumptions which are validated in Section 4; namely, the curve $y(x)$ should emanate from the point $x^0 = -a$, $y^0 = 0$; this requirement leads to a singularity: $y' \to -\infty$ when $x \downarrow -a$. As a result, boundary conditions (3.4) for Eq. (3.5) are reduced to the form

$$y(-a) = 0, \quad x^f + a\sin\alpha(x^f) = x_c^f, \quad y^f - a\cos\alpha(x^f) = y_c^f \tag{3.6}$$

where $y^f = y(x^f)$, and the quantities $\sin\alpha$ and $\cos\alpha$ are defined in terms of y' and s' according to the fifth and sixth relations of system (2.1). It is assumed that the solution of Eq. (3.5) can be presented in the form

$$y = y(x, y^0, x^f, y^f), \quad y(-a) = y^0 = 0, \quad y(x^f) = y^f \tag{3.7}$$

where x^f and y^f are constants to be determined from the last two conditions of (3.6). This procedure is extremely lengthy and provides only the formal possibility of solving variational problem (3.4). Note the extraordinary complexity of equation (3.6). Its integration and the satisfaction of the boundary conditions according to relations (3.6) and (3.7) are extremely difficult. All this means that the effective plotting of the brachistochrone requires appropriate transformation of the variational problem (3.3), (3.4), i.e., reduction to more convenient variables (see below).

4. Modification of the variational problem of the fastest motion of the disc centre

In order to simplify the solution of the brachistochrone problem for a disc, we will use the structural properties of the function Φ (3.2), (3.3) and boundary conditions (3.4). These properties stem from the expressions for the kinetic and potential energy, taking into account relations (2.1), (2.2), (2.4) and (2.5)

$$T_c = \frac{1}{2}m^* s_c'^2 \dot{x}_c^2, \quad V_c(y_c) = -mgy_c, \quad s_c'^2 = 1 + \left(\frac{dy_c}{dx_c}\right)^2, \quad y_c = y_c(x_c) \tag{4.1}$$

The curve $y_c(x_c)$ determines the trajectory of motion of the disc centre C. The variational problem takes the traditional (modified) form for plotting the classical brachistochrone

$$\theta^f = \sqrt{\frac{m}{m^*}2gt_f} = \int_0^{x_c^f} \frac{\sqrt{1+y_c'^2}}{\sqrt{y_c}}dx_c \to \min_{y_c}$$

$$y_c(0) = 0, \quad y_c(x_c^f) = y_c^f, \quad s_c = \int_0^{x_c} \sqrt{1+y_c'^2(\sigma)}\,d\sigma \tag{4.2}$$

According to Section 1, the solution of problem (4.2) is defined by a cycloid[1–3]

$$x_c = R_c(\xi - \sin\xi), \quad y_c = R_c(1-\cos\xi), \quad R_c = \frac{x_c^f}{\xi^f - \sin\xi^f}$$

$$\xi = \left(\frac{m}{m^*}\right)^{1/2}\left(\frac{g}{R_c}\right)^{1/2} t, \quad R_c > 0, \quad 0 \le \xi \le \xi^f \le 2\pi$$

$$\chi_c^f \equiv \frac{x_c^f}{y_c^f} = \frac{\xi^f - \sin\xi^f}{1-\cos\xi^f}, \quad 0 < \chi_c^f < \infty, \quad \xi^f = \xi^f(\chi_c^f) \tag{4.3}$$

The family of cycloids (x_c, y_c) for all (x_c^f, y_c^f), i.e., $R_c > 0$, $0 \le \xi \le 2\pi$, forms a field of extremals (see Fig. 2). For values of ξ_c^f and R_c corresponding to relations (4.3), the curve $y_c(x_c)$ defines the optimum solution of variational problem (4.2) in terms of the coordinates of the disc centre x_c, $y(x_c)$.

Thus, let the required optimum cycloid be plotted (see formulae (4.3) and Fig. 2). The response time t_f corresponding to the specified point (x_c^f, y_c^f) is calculated by means of a formula similar to formula (1.3), with the aid of expression (4.2) for θ^f, i.e.,

$$\xi = \sqrt{\frac{mg}{m^* R_c}}t \equiv \nu t, \quad 0 \le t \le t_f; \quad t_f = \left(\frac{mg}{m^* R_c}\right)^{-1/2}\xi^f, \quad 0 < \xi^f \le 2\pi, \quad R_c > 0 \tag{4.4}$$

Theorem. The solution of the classical brachistochrone problem for a disc, on the assumption of regular rolling without separation and slip, is defined by the modified curve $y(x)$ according to formulae (2.1) and (4.3) and has the form (Fig. 4)

$$x(\xi) = x_c(\xi) - a\sin\alpha_c(\xi), \quad \sin\alpha_c = \cos\frac{\xi}{2}, \quad \xi = \nu t$$

$$y(\xi) = y_c(\xi) + a\cos\alpha_c(\xi), \quad \cos\alpha_c = \sin\frac{\xi}{2} \tag{4.5}$$

From relations (4.5) we have the following values, adopted intuitively in Section 3:

$$x(0) = -a, \quad y'(0) = -\infty$$

The brachistochrone $y(x)$ is represented in Fig. 4 by the continuous bold curve. Its coordinates for the specified value of the parameter ξ, i.e., the time t, are defined as the sum of the coordinates of the radius vector of the point of the cycloid (x_c, y_c) and a vector **a** (unfixed) orthogonal to the cycloid and rotated by an angle α_c (see Fig. 4). In other words, the brachistochrone is the envelope of a family of circles, the centres of which move along the cycloid.

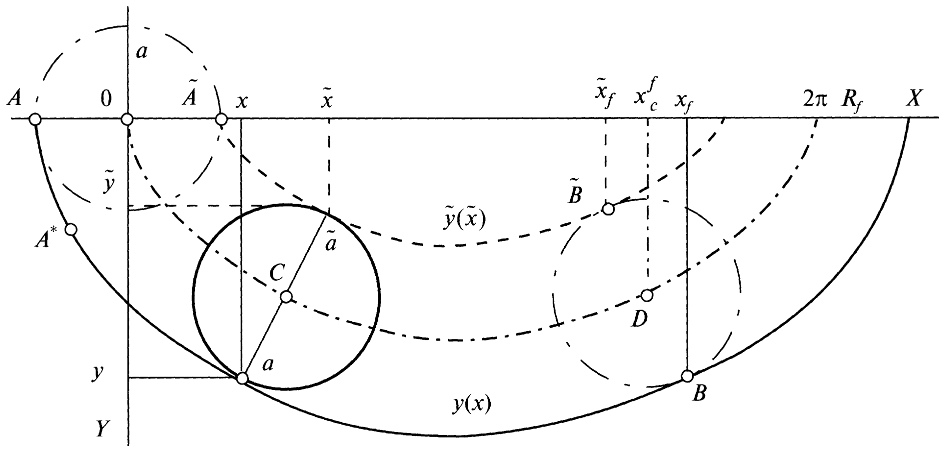

Fig. 4.

Note that the modified brachistochrone for the disc can be artificially realized when it is "rolling" along the curve $\tilde{x}(\xi)$, $\tilde{y}(\xi)$ connecting the points $\tilde{A}$ and $\tilde{B}$ (see Fig. 4, the dashed curve above and to the right of $y_c(x_c)$). With this motion, the disc rotates in the reverse direction (anticlockwise), and the constraint or normal pressure of the disc on the curve $\tilde{y}(\tilde{x})$ is created by additional devices.

Using expressions (4.5) for $x(\xi)$ and $y(\xi)$, taking into account relations (4.3) for $x_c(\xi)$ and $y_c(\xi)$, we can obtain the required kinematic characteristics of the brachistochrone for the disc. We have the expressions

$$y' = \frac{dy}{dx} = \text{tg}\alpha = \text{ctg}\frac{\xi}{2}, \quad s' = \frac{ds}{dx} = \sin^{-1}\frac{\xi}{2}, \quad \xi = \nu t$$

$$\kappa = -\left(a + 4R_c \sin\frac{\xi}{2}\right)^{-1}, \quad 0 < \xi < 2\pi, \quad 0 > \kappa > -a \tag{4.6}$$

from which it follows that the quantities y' and s' are unbounded, and $(1 + a\kappa) \to +0$ when $\xi \to +0$.

5. Analysis of the conditions for regular rolling of a disc along a brachistochrone

We will examine the equations of motion of the centre of mass of a disc, rolling along a brachistochrone, in the form of Newton's equations. We will differentiate expressions (2.2) with respect to t for $\dot{x}_c$, $\dot{y}_c$; by Newton's second law, we obtain equations for the coordinates of the centre of mass of the disc (Fig. 1)

$$m\ddot{x}_c = N\sin\alpha + F\sin\alpha, \quad \ddot{x}_c \equiv \ddot{x}(1 + a\kappa) + \kappa'\dot{x}^2$$

$$m\ddot{y}_c = mg - N\cos\alpha + F\sin\alpha$$

$$\ddot{y}_c \equiv \ddot{x}(1 + a\kappa)y' + \dot{x}^2(y''(1 + a\kappa) + ay'\kappa') \tag{5.1}$$

It must be assumed that the function $y(x)$ is known according to the penultimate equation of system (4.5), i.e., the quantities κ, κ', y', y'' and $\sin\alpha$, $\cos\alpha$ as functions of the argument x are known. The dependence of the variable $x(t)$ is determined explicitly either by integration of Cauchy's problem for Eq. (2.6) with the conditions $x(0) = -a$, $\dot{x}(0) = 0$, or by using expressions (4.3) to (4.5). Note that the second derivative $\ddot{x}$ in Eq. (5.1) is eliminated using expression (2.6).

The unknown forces of normal pressure N and sliding friction F (Fig. 1) are determined uniquely from the linear equations (5.1)

$$N = m(\ddot{x}_c\sin\alpha - (\ddot{y}_c - g)\cos\alpha), \quad F = m(\ddot{x}_c\cos\alpha + (\ddot{y}_c - g)\sin\alpha) \tag{5.2}$$

taking into account the expressions for $\ddot{x}_c$ and $\ddot{y}_c$ (5.1). The force of rolling friction[3] is ignored. The condition of regular rolling consists of checking the inequalities

$$N > 0, \quad |F| < k_f N \tag{5.3}$$

where k_f is the coefficient of dry friction.[3,5,6]

Analysis of expressions (5.2) indicates that, at the initial instant of time $t = 0$, the normal pressure $N = 0$, because $\ddot{x}_c(0) = \cos\alpha(-a) = 0$, and its value then begins to increase, and, when $\alpha \approx 0$ is reached, $N \approx mg$ is obtained. The force of sliding friction $F = 0$ at the initial stage of the motion. Thus, at the start of motion, strict inequalities (5.3) are not satisfied, and additional artificial approaches are required to realize regular rolling of the disc. For example, a model of "absolutely rough" contact surfaces can be created.[3,7–11]

Strict formulation of the brachistochrone problem under the additional condition of regular rolling must be considered taking into account inequalities (5.3), which makes its solution considerably more difficult. An "approximate" solution with a fairly high value of the coefficient of dry friction k_f will be the part of the brachistochrone A^*B that satisfies conditions (5.3) (see Fig. 4). Here, the disc can be brought to a state of translational-rotational motion, corresponding to point A^* of the ideal theoretical brachistochrone (4.5). The corresponding instant of time $t^* > 0$ can be determined from the equation $|F| = k_f N$, i.e.,

$$\left|\ddot{x}_c\cos\alpha + (\ddot{y}_c - g)\sin\alpha\right| = k_f(\ddot{x}_c\sin\alpha - (\ddot{y}_c - g)\cos\alpha) > 0 \quad (5.4)$$

Calculation of t^* requires mathematical modelling with a specified value of the friction coefficient k_f, the radius of the disc a, the ratio m/m^* and the parameters x_c^f, y_c^f.

Note that all linear quantities of the problem can be normalized on $a > 0$, i.e., in the initial system it can be assumed that $a = 1$. Moreover, as unit time t we can take the quantity $\nu^{-1} = \sqrt{\frac{a}{g}}$ (or $\nu^{-1} = \left(\frac{a}{g}\right)\left(\frac{m^*}{m}\right)^{1/2}$), i.e., we can introduce the argument $\tau = \nu t$ and assume that $g = 1$ (or $\nu = 1$) in the equations.

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